Reprint ISSN 0973-9424

International Journal Of Mathematical Sciences And Engineering Applications

(IJMSEA)

www.ascent-journals.com

International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 15 No. II (December, 2021), pp. 51-57

SOME THEOREMS ON MEASURABLE MULTIVALUED MAPPING

RAJ KUMAR SONI
Dept. of Mathematics,
Govt. Girls P.G. College, Rewa (M.P.), 486001, India
Affiliated to A.P.S. University, Rewa 486003, India

Abstract

In this paper, we have tried to investigate some new measurable multivalued function based on the concept of Souslin type as well as souslin operations.

1. Introduction

After the appearance of the book of Lusin [9] in 1930, who first attempted to investigate the measurability of multivalued function as well as existence of its selector. The major breakthrough in this direction began in 1965 due to JACOBS [6] and thereafter many eminent persons like CASTAING [3], HIMMELBERG [5], VAN-VLECK [5], RYLL-NARDZEWSKI [8], LEESE [10] tried to obtain the measurability conditions of multivalued mapping under different conditions on a various type of spaces both measurable as well as topological space. The main target of all these authors use to be find out the existence of measurable selector for the multivalued as well as implicit function theorem and lifting problem because such problem used to arise frequently in control theory, Mathematical Economics and probability theory in Statistics.

Key Words : *Multifunction of Souslin type, Souslin Operation.*

In 1974, S. J. LEESE [10] defined the Souslin type multivalued function and has proved the existence of measurable selector of such multifunctions. The class of multifunction is stable with respect to the usual operation of analysis, its name was given because of the parallel between this theory and the classical theory of Souslin sets.

In this paper we have tried to investigate some new measurable multivalued functions based on the concept of Souslin type as well as Souslin operation.

In section 2, we have given some necessary definitions and notations as well as stated some properties of multivalued mapping without proof but oftenly required in our main results.

2. Preliminaries

Definition 2.1 : Let T and X be any two sets, then a mapping $\Gamma : T \to X$ which assigns to each point t in T, a set $\Gamma(t)$ of points in X is called a multifunction.

A selector for Γ is a function f from T into X such that $f(t) \in \Gamma(t)$ for each $t \in T$.

Let M be the σ-algebra of subset of T, then multifunction Γ is said to be measurable if for any closed subset B of X, the set

$$\Gamma^{-1}(B) = \{t \in T : \Gamma(t) \cap B \neq \phi\} \in M.$$

Equivalently,

$$\begin{aligned} \Gamma^{-1}(G) &= \{t \in T : \Gamma(t) \subset G\} \in M \\ &= T \cap \{t \in T : \Gamma(t) \subset G\} \in M \end{aligned}$$

for every open subset G of X.

Definition 2.2: The graph of multifunction is denoted by $Gr(\Gamma)$ and defined by-

$$Gr(\Gamma) = \{(t, x) \in T \times X : x \in \Gamma(t)\}.$$

If Γ is upper semi continuous implies $Gr(\Gamma)$ is closed.

Definition 2.3 : We shall make use of the souslin operation, which is fully discussed by FELIX HAUSDORFF in “Set Theory” and K. KURATOWSKI in “Topology part I”.

let $\{A_{\sigma_1,\cdots,\sigma_n}\}$ be a complete collection of sets in a given space, indexed by the set of all finite sequences $\sigma_1,\cdots,\sigma_n$ of positive integers. Then the set

$$A = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} A_{\sigma_1,\cdots,\sigma_n}.$$

The union being taken over the collection of all infinite sequence σ of positive integer; is said to be obtained from the collection $\{A_{\sigma_1,\cdots,\sigma_m}\}$ by the souslin operation.
If the set $\{A_{\sigma_1},\cdots,\sigma_n\}$ belong to a given class N of sets then A will be said to belong to the class souslin $-N$. We shall use the notation-

$$A = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} A_{\sigma/n}$$

where σ/n denotes the finite sequence $\sigma_1,\cdots,\sigma_n$.
The measurable space T will be said to admit the Souslin operation if every subset formed in this way from measurable sets is measurable.
Definition 2.4 : Let T be a measurable space and X a topological space. Then a multifunction $\Gamma : T \to X$ is said to be of Souslin type if there exist a Polish space P, a measurable multifunction $\Omega : T \to P$ with closed values, and a continuous mapping $\phi : P \to X$ such that, for all $t, \Gamma(t) = \phi(\Omega(t))$.
A multifunction of Souslin type is necessarily be measurable.
R will denote the class of all set $A \times B$ where A is measurable set in T and B is closed set in X. We shall need the following Lemma.
Lemma 2.5 : Let X_1 be a topological space and K^* the class of sets which are closed and compact in X_1. Let Y be a Souslin-K^* subspace of X_1. Then if T is a measurable space which admits the Souslin operation, and π_1, is the canonical projection from $T \times Y$ into T, then $\pi_1(A)$ is measurable for every Souslin-R set A in $T \times Y$.
Lemma 2.6 : Let X, Y be topological spaces and $\psi : X \to Y$ is a continuous mapping. Then if T is a measurable space and $\Gamma : T \to X$ is a multifunction of Souslin type, so is the multifunction $t \to \psi(\Gamma(t))$.

3. Main Result

Theorem 3.1 : Let T be a topological space with Borel σ-algebra $B(T)$ of open subset of T, and X be a topological space. Then a multifunction $\Gamma : T \to X$.

Γ is measurable if there exist a metrizable compact space P, a usc multifunction Ω : $T \to P$ with closed values and a continuous mapping $\phi : P \to X$ such that $\forall \;\; T, \Gamma(t) = \phi(\Omega(t))$.

Proof : Since P is metrizable compact space, so it has a countable base.

Therefore P is Polish space (i)

Moreover since Ω is usc-multifunction from T into P, then for every closed set $B \subset P$,

$\Omega^{-1}(B) = \{t : \Omega(t) \subset P - B\}$is open

$$\Rightarrow \Omega^{-1}(B) \in B(T) \qquad (ii)$$

$\therefore$ Ω is measurable.

By using definition 2.4 the multifunction Γ is of Souslin type.

$\Rightarrow \Gamma$ is measurable because every multifunction of Souslin type is measurable.

Theorem 3.2 : Let T be a topological space with Boral σ-algebra $B(T)$ of open subsets of T and X be a topological space. If (Γ_n) is a sequence of a multifunctions defined from T into X, and there exit a metrizable compact spaces P_n, usc-multifunction $\Omega_n : T \to P_n$ with closed values and continuous mappings, $\phi_n : P_n \to X$ s.t. $\forall \; t \in T, \Gamma_n(t) = \phi_n(\Omega_n(t))$. Then the multifunction $\Gamma : T \to X$ defined by $\Gamma(t) = \bigcup\limits_{n=1}^{\infty} \Gamma_n(t)$ for each t is measurable.

Proof: From Theorem 3.1 each space P_n are Polish and each Ω_n are measurable.

$\therefore$ $\sum\limits_{n=1}^{k} P_n = P(let)$ will be the Polish space.

Let $\phi : P \to X$ be the continuous mapping which coincides with ϕ_i on P_i.

Now we define,

$$\Omega(t) = \sum_{n=1}^{\infty} \Omega_n(t) \;\; \forall \;\; t \in T$$

$\Rightarrow$ $\Omega(t)$ is closed values and measurable.

$$\therefore \;\; \Gamma(t) = \bigcup_{n=1}^{\infty} \Gamma_n(t) = \bigcup_{n=1}^{\infty} \phi_n(\Omega_n(t)) = \phi(\Omega(t))$$

$\Rightarrow$ multifunction Γ is of Souslin type.

Hence Γ is measurable multifunction.

Theorem 3.3 : Let T be a topological space with Borel σ-algebra of open subsets of T. Which admits the Souslin operation, (X_i) is a sequence of topological space and

(Γ_i) is a sequence of multifunction from T into X_i respectively such that there exist metrizable compact space P_i, and usc multifunctions $\Omega_i : T \to P_i$ with closed values and continuous function $\phi_i : P_i \to X_i$ respectively s.t.

$\Gamma_i(t) = \phi_i(\Omega_i(t))$ $\forall$‘ $t \in T$. Then the multifunction -

$\Gamma : T \to X_1 \times X_2 \times \cdots$ is measurable for each t defined by

$$\Gamma(t) = \Gamma_i(t) \times \Gamma_2(t) \times \cdots .$$

Proof : From Theorem 3.1 each space P_i are polish space and each usc multifunction Ω_i are measurable multifunction.

The corresponding closed valued multifunction

$$\Omega_i : T \to P_i, i = 1, 2, \cdots$$

and continuous mapping are

$$\phi_i : P_i \to X_i, \quad i = 1, 2, \cdots .$$

Now we define

$$\Omega(t) = \prod_{i=1}^{\infty} \Omega_i(t)$$

and $P = P_1 \times P_2 \times \cdots$

From N. Bourbaki [2], P is Polish.

Let $G = G_1 \times G_2 \times \cdots$ be any open set in P. Then it is easily can be seen that

$\Omega^{-1}(G) = \bigcap_{n=1}^{\infty} \Omega_i^{-1}(G_i)$ is measurable set.

Since any open set H in P is a countable union of open sets H_i $(i = 1, 2, \cdots)$ then

$\Omega^{-1}(H) = \bigcup_{n=1}^{\infty} \Omega_i^{-1}(H_n)$ is measurable set.

$\Rightarrow$ Ω is a measurable multifunction from Theorem 3 of Robertson [12]. Again we define a mapping ϕ from P into $X_1 \times X_2 \times \cdots$ by

$$\phi(P_1, P_2, \cdots) = (\phi_1(P_1), \phi_2(P_2), \cdots).$$

$\Rightarrow$ ϕ is a continuous mapping.

Since $\Gamma(t) = \prod_{i=1}^{\infty} \Gamma_i(t)$ for each t

$\Rightarrow$ $\Gamma(t) = \prod_{i=1}^{\infty} \phi_i \Omega_i(t).$

$\Rightarrow \quad \Gamma(t) = \phi(\Omega(t))$.

$\Rightarrow$ Multifunction Γ is of a souslin type.

Hence Γ is measurable multifunction.

Theorem 3.4 : Let T be a topological space with Borel σ-algebra $B(T)$ of open subset of T which admits the souslin operation, V be a topological vector space and Γ_1, Γ_2 are multifunctions from T into V such that there exist metrizable compact spaces $P_i, i = 1, 2$; usc multifunctions $\Omega_i : T \to P_i$ and continuous mapping $\phi_i : P_i \to V$ s.t. $\Gamma_i(t) = \phi_i(\Omega_i(t))$ for $i = 1, 2$ and each $t \in T$. Then the multifunction $\Gamma : t \to \Gamma_1(t) + \Gamma_2(t)$ is measurable.

Moreover, if α is any measurable scalar valued function on T the multifunction, $t \to \alpha(t)\Gamma_1(t)$ is also measurable.

Proof: From Theorem 3.3, the multifunction $\Gamma : t \to \Gamma_1(t) \times \Gamma_2(t)$ is measurable from T into $V \times V$.

We know that the mapping $\phi : V \times V \to V$ defined by

$\phi(x, y) = x + y$ is continuous.

So that by Lemma 2.6, $\Gamma(t) = \phi(\Omega(t))$ is of a Souslin type, where $\Omega(t) = \Omega_1(t) \times \Omega_2(t)$.

Thus Γ is a measurable multifunction.

For the second part of Theorem, by the Theorem 3.3, the multifunction $t \to \{\alpha(t)\} \times \Gamma_1(t)$ is measurable and we know that the mapping $\phi : (\lambda, x) \to \lambda x$ is continuous, where λ is any arbitrary scalar.

Then we have

$$\phi(\{\alpha(t)\} \times \Gamma_1(t)) = \alpha(t)\Gamma_1(t)$$

$\Rightarrow$ the multifunction $t \to \alpha(t)\Gamma_1(t)$ is of souslin type.

$\Rightarrow$ the multifunction $t \to \alpha(t)\Gamma_1(t)$ is measurable.

References

[1] Aumann R. J., Integral of set value function, J. Math. Anal. Appl., 12 (1965), 1-12.

[2] Bourbaki N., General Topology, Part -2, Addition-Wesley, Reading, Mass, (1966).

[3] Castaing C., Sur Les multifunction measurables, Rev. Francaise Informat. Research Operationebel, 1 (1967), 91-126.

[4] Fillippov A. F., On certain questions in the theory of optimal control, SIAM J. Control Sec. A1 (1962-63), 76-84.

[5] Himmelberg C. J. and Van-Vleck F. S., Selection and implicit function theorems for multifunction with souslin graph, Bull. Acad. Polon. Sci. Sec. Sci. Math. Astronom. Phys., 19 (1971), 911-916.

[6] Jacobs M. Q., Measurable multivalued mapping and Lusin's theorem, Trans. Amer. Math. Sco., 134 (1968), 471-481.

[7] Kuratowski K., Topology, Vol.I, New-York, London, Warszawa (1966).

[8] Kuratowski K. and Rvll-Nardzewski C., A general theorem on selector, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., 13 (1965), 397-403.

[9] Lusin N., Lecons Sur les Ensembles Analyliques et Leurs Appl. Chelsea, New York, (Second Editiion) (1972), (First edition was 1930).

[10] Leese S. J., Multifunction of Souslin tyipe, Bull. Austral Math. Soc., 11 (1974), 395-411.

[11] Moshane E. J. and Warfield R. B., On Fillippov's implicit function lemma, Proc. Ames. Math. Soc., 18 (1967), 41-47.

[12] Robertson A.P., On measurable selection, Proc.Roy. Soc.Edinburgh, Ser. A 72 (1972-73), 1-7 (1974).

[13] Von-Neumann J., On rings on operators reduction theory, Anal. Math., 50 (1949), 401-485.